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Extensions of Rectangular Band Anti-congruence in Semigroup with Apartness[1]

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Abstract

The setting of this research is Bishop's constructive mathematics. Let q be an anticongruence on a semigroup S with apartness. For q we say that it is an rectangular band anticongruence if S/q is a rectangular band. In paper [12] a construction of the maximal rectangular band anticongruence on a semigroup is described. This paper is continuation of that research. Let T be a detachable subsemigroup of a semigroup S. Necessary and sufficient conditions for which any rectangular band anticongruence on T can be extended to rectangular band anticongruence on S are given. Every rectangular band anticongruence on T can be extended to a rectangular band anticongruence on S if and only if the maximal rectangular band anticongruence on T can be extended to a rectangular band anticongruence on S.

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1 Introduction

1.1 Setting and motivation

Our setting is Bishop's constructive mathematics (in sense books [1], [2] and [6]), mathematics developed with Constructive Logic (or Intuitionistic Logic

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([13])) - logic without the Law of Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive Logic. Precisely, in Constructive Logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal Logic. In Constructive Logic 'Weak Law of Excluded Middle'$\neg P \vee \neg\neg P$ does not hold, too. It is interesting, in Constructive Logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove it without 'the crazy axiom'. Any notion in Bishop's constructive mathematics has positive defined symmetrical pair since Law of Excluded Middle does not hold in Constructive Logic. Our intention is development of these symmetrical notions and their compatibility with so-called the 'first notions' in Semigroup Theory. As the first, semigroup is equipped with diversity relation compatible with the equality, and, the second, the semigroup operation is totaly extensional and strongly extensional function. Symmetrical relations to Green's relations are also interesting in Constructive Algebra.

1.2 Set with apartness

Let $(S, =, \neq)$ be a set (in the sense of books [1], [2], [6] and [13]), where "$=$" is an equality and "$\neq$" is a binary relation on S which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z$$

called *diversity relation* on S. Following Heyting, if the relation satisfies the following implication

$$x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z),$$

we say that it is an *apartness*. Let Y be a subset of S and let $x \in S$. Following Bridges, by $x \bowtie Y$ we denote $(\forall y \in Y)(y \neq x)$ and by Y^C we denote subset $\{x \in S : x \bowtie Y\}$ - the strong complement of Y in S ([13]). The subset Y of S is *strongly extensional* ([13]) in S if and only if $y \in Y \Longrightarrow y \neq x \vee x \in Y$, $x \in S$. For a subset Y of S we say that it is a *detachable* subset of S if and only if $(\forall x \in S)(x \in Y \vee \neg(x \in Y))$.

Examples I: (1) Let $\wp(S)$ be power-set of set S. If we for subsets A, B of S define $A \neq B$ if and only if $(\exists a \in A)\neg(a \in B)$ or $(\exists b \in B)\neg(b \in A)$, then the relation "$\neq$" is a diversity relation on $\wp(S)$ but it is not an apartness.

(2) ([6]) The relation "$\neq$" defined on the set $\mathbf{Q}^N$ by

$$f \neq g \Longleftrightarrow (\exists k \in N)(\exists n \in N)(m \geq n \Longrightarrow |f(m) - g(m)| > k^{-1})$$

is an apartness on $\mathbf{Q}^N$. $\diamond$

Let S be a set with apartness and let α, β be relations on S. The *filed product* ([7], [8], [10], [11]) of α and β is the relation defined by

$$\beta * \alpha = \{(x,z) \in S \times S : (\forall y \in S)((x,y) \in \alpha \vee (y,z) \in \beta)\}.$$

For $n \geq 2$ let ${}^{n}\alpha = \alpha * ... * \alpha$ (n factors). Put ${}^{1}\alpha = \alpha$. By $c(\alpha)$ we denote the intersection $c(\alpha) = \bigcap_{n \in N} {}^{n}\alpha$. The relation $c(\alpha)$ is a cotransitive relation on S, by Theorem 0.4 of paper [11], called *cotransitive internal fulfillment* of relation α.

A relation q on S is a *coequality relation* on S ([7]-[12]) if and only if

$$q \subseteq \neq, \;\; q^{-1} = q, \;\; q \subseteq q * q.$$

For equality e and coequality q we say that they are *compatible* if and only if

$$e \circ q \subseteq q, \;\; q \circ e \subseteq \mathrm{q}.$$

In this case we can construct the following factor-sets: $S/(e,q) = \{ae : a \in S\}$, $S/(q^C, q) = \{aq^C : a \in S\}$ and $S/q = \{aq : a \in S\}$.

1.3 Semigroup with apartness

Let $S = (S, =, \neq, \cdot)$ be a semigroup with apartness and where the semigroup operation is strongly extensional in the next sense

$$(\forall a,b,x,y \in S)((ay \neq by \Longrightarrow a \neq b) \wedge (xa \neq xb \Longrightarrow a \neq b)).$$

A subset T of S is a *consistent subset* of S ([4]) if and only if $(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T)$. Let T be a consistent subset of a semigroup S. T is a *filter* of S if T is a subsemigroup of S ([4]).

Example II: Let S be the set Let S be the set

$$\left\{ \begin{pmatrix} 0\,0 \\ x\,x \end{pmatrix}, \begin{pmatrix} 0\,x \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 0\,x \end{pmatrix}, \begin{pmatrix} x\,x \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} : x \in R \wedge 0 \leq x \leq \frac{1}{2} \right\}.$$

The operation on S is the usual matrix multiplication. Then S is a semigroup with apartness. The set $Q = \{f \in S : f \neq 0\}$ is a consistent subset of S. $\diamond$

Let q be a coequality relation on semigroup S. For q we say that it is an *anticongruence* on S ([7]-[12]) if and only if

$$(\forall a,b,x,y \in S)((ax, by) \in q \Longrightarrow (a,b) \in q \vee (x,y) \in q).$$

This is equivalent with the following:

$$(\forall u,x,y \in S)(((ux,uy) \in q \Longrightarrow (a,b) \in q) \wedge ((xu,yu) \in q \Longrightarrow (x,y) \in q)).$$

If q is anticongruence on semigroup S, then the strong complement q^C of q is a congruence on the semigroup S compatible with q. In this case we can construct semigroups $S/(q^C,q) = \{aq^C : a \in S\}$ and $S/q = \{aq : a \in S\}$.

Example III: Let T be a strongly extensional subset of a semigroup S. Then:

(1) If T is a consistent subset of S, then the relation q on S, defined by

$$(a,b) \in q \Longleftrightarrow a \neq b \wedge (a \in T \vee b \in T),$$

is an anticongruence on S;

(2) The relation q on S defined by

$$(a,b) \in q \Longleftrightarrow (\exists x,y \in S^1)((xay \in T \wedge xby \bowtie T) \vee (xby \in T \wedge xay \bowtie T))$$

is an anticongruence on S. $\diamond$

Semigroups with apartnesses were defined and studied for the first time by A. Heyting. P. T. Johnstone, J. C. Mulvey, F. Richman, R. Mines, D. A. Romano, W. Ruitenburg, A. S. Troelstra and D. van Dalen also have some results in this field. There are more general problems on semigroup with apartness in Constructive Algebra. In this paper we present a construction of a coequality relation q on a semigroup S with apartness by using of left and right principal consistent subsets of semigroups such that q is a rectangular band anticongruence on S. Besides, we will describe classes of q.

Lemma 1.1 ([4], Theorem 1.24) *The following conditions for a semigroup S are equivalent:*
(1) $(\forall a,b \in S)((a,ab) \bowtie q \wedge (a,ba) \bowtie q)$*;*
(2) $(\forall a,b \in S)((a,aba) \bowtie q)$*;*
(3) $(\forall a,b,c \in S)((a,a^2) \bowtie q \wedge (abc,ac) \bowtie q)$*.*

Recall that a semigroup S is called *band* if $a^2 = a$, for all $a \in S$. We have the following:

Lemma 1.2 ([3], Lemma 1) *Relation θ on band S, defined by*

$$(a,b) \in \theta \Longleftrightarrow a \neq ab \vee a \neq ba$$

is an anti-order relation on S.

So, band S is supplied with compatible pair of relations: order "$\leq$ " and anti-order "θ " defined by $x \leq y$ iff $x = xy = yx$, and $x\theta y$ iff $x \neq xy$ or $x \neq yx$, for every x,y in S. Let us note that if $(S, =, \neq, \cdot)$ is a band, then $(S, =, \neq, \cdot, \leq, \theta)$ is not an ordered semigroup, in general (unless in the special case when the multiplication on S is commutative). If q^C is a band congruence on semigroup $S = (S, =, \neq, \cdot)$, i.e. if $(\forall a \in S)((a, a^2) \bowtie q)$, then we say that q is a band anticongruence on S. If q^C is a left zero and right zero band congruence on S ([3], [4]), i.e. if

$$(\forall a, b \in S)((a, ab) \bowtie q \wedge (a, ba) \bowtie q)$$

we say that q is a *rectangular band anticongruence* on S.

1.4 Goal of this paper

In the classical Semigroup Theory, there several papers in which studied the Green relations. For example, in paper [5]. Various characterizations of the semigroups by Green's relations have been investigated by A. H. Cliford and G. B. Preston in their well-known book and in Bogdanicić - Ćiric's book [4]. Since in Constructive logic the 'Law of Excluded Middle' is not valid, in Bishop's constructive algebra symmetrical relations to Green's relations are also interesting. In his articles [7]-[11] the author researches some special relations and subsets of semigroup with apartness generated by strongly extensional (left or right) consistent principal subsets. Particularly, in article [11] the author studied relations $c(s \cup s^{-1})$ and $c(s \cap s^{-1})$ and their classes, where $(a, b) \in s \iff b \in C_{(a)} = \{x \in S : x \bowtie SaS\}$. Some properties of relation $c(s)$ are given in article [11]. In article [8] the following results are given: Let a be an arbitrary element of semigroup $(S, =, \neq, \cdot)$. Then the set $L_{(a)} = \{x \in S : x \bowtie Sa\}$ is a right consistent subset of S. The relation l on S, defined by $(a, b) \in l \iff b \in L_{(a)}$ is (Theorem 2.2) a consistent relation on S and the relation $c(l)$ is (Theorem 2.3) a quasi-antiorder on set S, left compatible with the semigroup operation, such that that left class $A(a)$ is the maximal strongly extensional right consistent subset of S with $a \bowtie A(a)$, and right class $B(a)$ is the maximal strongly extensional left ideal of S with $a \bowtie B(a)$. In article [10] relation $(a, b) \in p \iff b \in L_{(a)} \wedge a \in L_{(b)}$ are investigated: the relation $c(p)$ is (Theorem 3) right zero band anticongruence on S (An anticongruence q on semigroup S is *right zero band anticongruence* if and only if the factor-semigroup $S/(q^C, q)$ is a right zero band.) and the class $A(a)$ is (Theorem 5) the maximal strongly extensional right consistent left ideal of S with $a \bowtie A(a)$. In article [3] we investigate band anticongruenece of ordered semigroup S under a pair od an order and an anti-order relations on S. In article [12] we investigate the following relation m, defined by

$$(a, b) \in l \iff b \in L_{(a)},\ \ (a, b) \in r \iff b \in R_{(a)},\ m = l \cap l^{-1} \cap r \cap r^{-1},$$

i.e. relation m on semigroup S defined by

$$(a,b) \in m \Longleftrightarrow b \in L_{(a)} \wedge a \in L_{(b)} \wedge a \in R_{(b)} \wedge b \in R_{(a)},$$

and relation $c(m)$ in particular.

Here, we continue our research of band anticongruence on semigroups with apartness. Let us remaind that a algebraic structure A has the (anti-congruence) congruence extension property if for any algebraic substructure $B \leq A$ and any (anti-congruence) congruence relation θ on B there exists a (anti-congruence) congruence relation ψ on A such that $\psi \cap (B \times B) = \theta$.

By results of paper [12], always there exists the maximal rectangular band anticongruence on S. In the section 3 of this paper necessary and sufficient conditions for which any rectangular band anticongruence on a detachable subsemigroup T of a semigroup S can be extended to rectangular band anti-congruence on S are given.

1.5 References

For undefined notions and notations of semigroup theory we refer to [4], [5] and of items in Constructive mathematics we refer to books [1], [2], [6], [13] and to papers [3], [7]-[12].

2 A Construction of the Maximal Rectangular Anti-congruence

This section we start with the following statement:

Lemma 2.1 ([12], Lemma 2) *Let q be a rectangular anticongruence on a semigroup S. For any element a in S, aq is a strongly extensional consistent ideal of S such that $a \bowtie aq$ and $a^2q = aq$, $abaq = aq$ and $abcq = acq$.*

The next three theorems are main results of paper [12].

Theorem 2.2 ([12], Theorem 5) *The relation $c(m)$ is a rectangular anticongruence on S.*

By an element a of a semigroup S and for $n \in N$ we introduce the following notations

$$A(a) = \{x \in S : (a,x) \in c(m)\}, \ \ A_n(a) = \{x \in S : (a,x) \in {}^n m\}.$$

By the following theorem we will present some basic characteristics of these sets:

Theorem 2.3 ([12], Theorem 6, Theorem 7) *Let a, b and c be elements of S. Then:*

(1) $A_1(a) = \{x \in L_{(a)} \cap R_{(a)} : a \in L_{(x)} \cap R_{(x)}\}$*;*
(2) $(\forall n \in N)(A_{n+1}(a) \subseteq A_n(a))$*;*
(3) $(\forall n \in N)(A_{n+1}(a) = \{x \in S : S = A_n(a) \cup A_1(x)\})$*;*
(4) $A(a) = \bigcap_{n\in N} A_n(a)$*;*
(5) $a \bowtie A(a)$*;*
(6) The set $A(a)$ is a strongly extensional consistent ideal of S;
(7) $A(a^2) = A(a)$*;*
(8) $A(aba) = A(a)$*;*
(9) $A(abc) = A(ac)$*;*
(10) Let a be an element of S. Then the set $A(a)$ is the maximal strongly extensional consistent ideal of S such that $a \bowtie A(a)$ and $A(a^2) = A(a)$, $A(aba) = A(a)$ and $A(abc) = A(ac)$.

Finally, we have assertion that the relation $c(m)$ is the maximal rectangular anti-congruence on S:

Theorem 2.4 ([12], Theorem 8) *The relation $c(m)$ is the maximal rectangular anti-congruence on a semigroup S with apartness.*

3 Extending Rectangular Band Anticongruence

Let S be a semigroup and T a subsemigroup of S. For an anti-congruence q on S, q restricted to T is denoted by $q_T = q \cap (T \times T)$.

Let S be a semigroup, T a subsemigroup of S and q_T anticongruence on T. If there exists an anticongruence q on S such that $q \cap (T \times T) = q_T$, then for q we say that it is an anticongruence *extension* of q_T from T to S.

Lemma 3.1 *If the maximal rectangular band anti-congruence $c(m_T)$ on T can be extended to a rectangular band anti-congruence on S, then $c(m_S)$ is an anti-congruence extension of $c(m_T)$ from T to S.*

Proof: Suppose that $c(m_T)$ on T can be extended to a rectangular band anticongruence q_S on S. Since $c(m_T)$ is the maximal rectangular band anticongruence on T, then $c(m_T) \supseteq c(m_S) \cap (T \times T)$. Moreover, $q_S \subseteq c(m_S)$ implies that

$$c(m_T) = q_S \cap (T \times T) \subseteq c(m_S) \cap (T \times T).$$

Thus, $c(m_T) = c(m_S) \cap (T \times T)$. $\diamond$

Let T be a subsemigroup of a semigroup $S = (S, =, \neq, \cdot)$. By an element a of S and an element b of T we introduce the following notations

$$A_S(a) = \{x \in S : (a, x) \in c(m_S)\}, \;\; A_T(b) = \{x \in T : (b, x) \in c(m_T)\}.$$

By the following we give some basic characteristics of these sets:

Theorem 3.2 *The maximal rectangular band anticongruence $c(m_T)$ on T can be extended to a rectangular band anticongruence on S if and only if, for any a of T, $A_T(a) = A_S(a) \cap T$ holds.*

Proof: (1) Suppose that the maximal rectangular band anticongruence $c(m_T)$ on T can be extended to a rectangular band anticongruence on S. By Lemma 3.1, $c_S(m)$ is an anticongruence extension of $c(m_T)$, i.e. $c(m_T) = c(m_S) \cap (T \times T)$. It is easily to check that for any a in T the following holds $A_T(a) = A_S(a) \cap T$. In fact: The inclusion $A_T(a) \subseteq A_S(a) \cap T$ is clear. Let $a \in T$ and $x \in A_S(a) \cap T$ be an arbitrary element. Then $a \in T$, $x \in T$ and $(a, x) \in c_S(m_S)$. Since $c(m_T) = c(m_S) \cap (T \times T)$, we have that $(a, x) \in c(m_S) \cap (T \times T) = c(m_T)$. So, $x \in A_T(a)$.

(2) Let, for any a in T, the following $A_T(a) = A_S(a) \cap T$ holds. Since the inclusion $c(m_S) \cap (T \times T) \subseteq c(m_T)$ is clear, we need only to prove the inclusion $c(m_T) \subseteq c(m_S) \cap (T \times T)$. Let (x, y) be an arbitrary element of $c(m_T)$. Then $x \in T$, $y \in T$ and $y \in A_T(x) = A_S(x) \cap T$. Therefore, $(x, y) \in c_S(m_S) \cap (T \times T)$. $\diamond$

Lemma 3.3 *Let S be a rectangular band. Then any anticongruence on any detachable subsemigroup of S can be extended to anticongruence on S.*

Proof: In this case, the relation "$\neq$" on semigroup S and in any subsemigroup T of S is the maximal rectangular band anticongruence. Thus, if q_T is an anticongruence on T, then the relation $q_S = q_T \cup (((S \backslash T) \times (S \backslash T)))$ is an anticongruence on S such that $q_T = T \cap q_S$. $\diamond$

Lemma 3.4 ([9], Lemma 2) *Let α and β be anticongruences on a semigroup S with apartness such that $\beta \subseteq \alpha$. Then the relation β/α on S/α , defined by $\beta/\alpha = \{(a\alpha, b\alpha) \in S/\alpha \times S/\alpha : (a, b) \in \beta\}$ is a anticongruence and $(S/\alpha)/(\beta/\alpha) \cong S/\beta$ holds.*

Lemma 3.5 *Let S be a semigroup, Q an anticongruence on $S/c(m_S)$. Then the relation q on S, defined by $(a, b) \in q$ if and only if $(ac(m_S), bc(m_S)) \in Q$, is an anticongruence on S.*

Proof follows immediately from definitions of relation Q and factor-semigroup $S/c(m_S)$. $\diamond$

Theorem 3.6 *Let T be a detachable subsemigroup of a semigroup S. Then every rectangular band anticongruence on T can be extended to a rectangular band anticongruence on S if and only if the maximal rectangular band anticongruence on T can be extended to a rectangular band anticongruence on S.*

Proof: Suppose the maximal rectangular band anticongruence $c(m_T)$ can be extended to a rectangular band anticongruence on S. By Lemma 3.1, the relation $c(m_S)$ is an anti-congruence extension of $c(m_T)$. Since $T/c(m_T) = T/(c(m_S) \cap (T \times T))$, we can consider $T/c(m_T)$ as a detachable subsemigroup of $S/c(m_S)$. By Lemma 3.4, every anticongruence Q_T on $T/c(m_T)$ can be extended to an anticongruence on $S/c(m_S)$. Since $q_T \subseteq c(m_T)$ for any rectangular band anticongruence q_T on T we have $T/q_T \cong (T/c(m_T))/(q_T/c(m_T))$ by Lemma 3.5. So, the relation $qT/c(m_T)$ is a rectangular band anticongruence on detachable rectangular band $T/c(m_T)$. Thus, by Lemma 3.3, there exists an anticongruence $q_S/c(m_T)$ on $S/c(m_S)$ which is an anticongruence extension of $q_T/c(m_T)$ from $T/c(m_T)$ to $S/c(m_S)$. Define a relation q on S on the following way: for any x,y in S, $(x,y) \in q \Longleftrightarrow (xc(m_S), yc(m_S)) \in q_S/c(m)$. By Lemma 3.5, q is an anticongruence on S. By definition of q, for a, b in T, $(a,b) \in q$ if and only if $(ac(m_T), b(m_T)) \in q_S/c(m_S)$. So, $(a,b) \in q_T$. It is shown that an anti-congruence extension of q from T to S.

Let a, b be arbitrary element of S and let (u,v) be an arbitrary element of q. Then

$$\begin{aligned}
(u,v) \in q &\Longrightarrow (u,a) \in q \vee (a,ab) \in q \vee (ab,v) \in q \\
&\Longrightarrow u \neq v \vee (ac(m_S), abc(m_S)) \in q_S/c(m_S) \vee ab \neq v \\
&\Longrightarrow u \neq a \vee ab \neq v \\
&\Longrightarrow (a,ab) \neq (u,v) \in q.
\end{aligned}$$

The assertion $(b,ab) \bowtie q$ follows on same way. So, the relation q on S is a rectangular band anticongruence on S. $\diamond$

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